Solution to Satisfiability problem by a complete Grover search with trapped ions

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The main idea in the original Grover search (Phys. Rev. Lett. **79**, 325 (1997)) is to single out a target state containing the solution to a search problem by amplifying the amplitude of the state, following the Oracle's job, i.e., a black box giving us information about the target state. We design quantum circuits to accomplish a complete Grover search involving both the Oracle's job and the amplification of the target state, which are employed to solve Satisfiability (SAT) problems. We explore how to carry out the quantum circuits by currently available ion-trap quantum computing technology.

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I. INTRODUCTION

Quantum algorithms, such as the well-known Shor's factoring algorithm [1] and Grover's search algorithm [2], have shown exponential and quadratic speed-up, respectively, in computation over classical counterpart, due to the capabilities exploiting the parallelism of quantum mechanics or interference effects. It is considered that the future quantum computers should be able to solve some classically intractable problems [1, 3], e.g., the nondeterministic polynomial-complete (NPC) problems [4, 5], among which the random Boolean K-Satisfiability (K-SAT) problem [6] is a central issue in computer science. Many proposals have so far focused on the solution of hard instances of K-SAT problems by the method of so-called DPLL algorithm [7], quantum adiabatic algorithm [8, 9, 10, 11], Grover's search algorithm [12], Hogg's algorithm [13], local search algorithms [14, 15, 16, 17], statistical mechanics approach [18], etc.

We will concentrate in the present paper on solution to K-SAT problems by Grover search algorithm. Our motivation is to show the possibility with Grover search finding answers to some solvable K-SAT problems. Although most K-SAT, even if $K = 3$, are NP problems and Grover search does not own exponential speed-up capability in solution, we will consider some solvable cases of K-SAT as examples. As we know from the original Grover's papers and other subsequent work, Grover search was only considered as a method to efficiently single out the answer states (or say, target state in the language of Grover search), whereas the job to find the target state is resorted to a black box named Oracle. To work out a realistic problem, however, we should have to consider how to take the Oracle's work. So it naturally arises a question: Is it possible to design a scheme with consideration of both the job by Oracle and the job of amplitude amplification?

We have noticed a recent publication [12] with quantum circuits for a complete Grover search including Oracle's job and amplification of target state, in which a three-variable (i.e., three-qubit in quantum computing (QC) treatment) SAT problem is solved by employing seven auxiliary qubits and tens of single and multiple quantum gates. Enlightening by this idea, we will design some simplified quantum circuits for solutions to some K-SAT problems, which enables us to work with trapped-ion QC.

The ion-trap system [19, 20, 21] favors QC owing to long coherence time of qubits, high efficiency of detection, and full controllability of operations. In the context of trapped-ion QC, there have been some schemes [22, 23] for implementation of Grover search by constructing multiqubit conditional operations based on Cirac-Zoller (CZ) gate [24] or Mølmer-Sørensen gate [25]. Experimentally, two-qubit Grover search has been carried out by $^{111}Cd^{+}$ [26], and eight ions have been confined stably in entanglement in the trap [27], which is the most qubits ready for QC among the currently available QC candidate systems. Most recently, Grover search was carried out by a relatively simple way [28] making use of collective states of trapped ions [29].

We will explore below the experimental feasibility of solving K-SAT problems by a complete Grover search with trapped ions, where the Grover search will be implemented either by sequences of single-qubit and multiqubit quantum

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gates based on the light-shift (LS) gates [30] or directly by multiqubit conditional phase flip (CPF) gates [31]. We will compare the two methods for implementation of our solutions. The outline of the paper is as follows. In Section II, the K-SAT problem and Grover search algorithm are briefly introduced. In Section III, we apply the complete Grover search to some satisfiable K-SAT problems by quantum circuits, which compared to in [12], is much simplified with reduced number of the auxiliary qubits and gating. Then we will discuss the experimental feasibility with trapped ions for implementation in Section IV. The last section is for our conclusion.

II. K-SAT PROBLEM AND GROVER SEARCH ALGORITHM

Let us first of all review the K-SAT problem briefly. As a paradigmatic example of a NPC problem, the well-known K-SAT problem is actually a combinatorial search problem in theoretical computer science [32]. Generally speaking, the K-SAT problem could be expressed by a logical configuration involving ξ Boolean variables b_i $(i = 1, 2, \cdots, \xi)$ with the values $0 = FALSE$ and $1 = TRUE$. A K-SAT formula F_K, which consists of m clauses $\{C_\mu\}_{\mu=1,2,\cdots m}$, could be written as [10],

$$F_K = C_1 \wedge C_2 \wedge C_3 \cdots \wedge C_m, \tag{1}$$

where $\wedge$ means logical AND gate, and each clause C_i contains a number of logical variables b_i or its negation $\bar{b}_i$ (i.e., logical NOT gate on b_i). These variables (b_i or $\bar{b}_i$) are connected with each other by $\vee$ (logical OR gate), and the maximal number of variables in each clause is K. For instance, $F_2 = (\bar{b}_2 \vee b_4) \wedge (\bar{b}_2 \vee b_3) \wedge (b_1 \vee b_3)$ is a 2-SAT formula with 3 clauses and 4 logical variables. A solution to F_K is to find the values of the logical variables satisfying all clauses simultaneously in order to make F_K be $TRUE$. In some cases, there is no solution to a SAT problem, which is called unsatisfiable, and in some other cases, there are possibilities to have multiple solutions to a SAT problem. In the present scheme, for simplicity, we will restrict our study, exclusively, on the formulae with only one solution.

The primary idea of Grover's algorithm in [2] is to boost the probability amplitude of the target state so that we could measure the target state with a considerably high probability. Generally speaking, provided that the initial state of the system has been prepared in an average superposition state $|\Psi_0\rangle = (1/\sqrt{N}) \sum_{i=0}^{N-1} |i\rangle$ ($N = 2^n$ with n the qubit number), Grover search can be depicted as the iterative operation $G = \hat{D}^{(n)} I_\tau$ (defined later) by at least $\pi\sqrt{N}/4$ times for finding the marked state $|\tau\rangle$ with an optimal probability [33, 34], where the quantum phase gate $I_\tau = I - 2|\tau\rangle\langle\tau|$ (with I being the identity matrix) plays an important role of *Selective-Inversion* (SI) to invert the amplitude of the target state, and the diffusion transform $\hat{D}^{(n)}$ defined as $\hat{D}^{(n)}_{ij} = 2/N - \delta_{ij}$ $(i, j = 1, 2, 3, \cdots\cdots N)$ is called *Inversion-About-Average* (IAA).

III. SOLUTION TO K-SAT PROBLEMS BY COMPLETE GROVER SEARCH

In this section, we will show how Grover search can be carried out to find a satisfiable solution to the K-SAT problem. For simplicity, we first consider a 2-SAT formula F_2 involving three clauses and two Boolean variables,

$$F_2 = (\bar{a} \vee \bar{b}) \wedge (a \vee b) \wedge a. \tag{2}$$

To solve F_2, we have to employ some basic quantum logical gates including Controlled-NOT(CNOT), AND, and OR, as depicted in Fig. 1 where the multiqubit CNOT gate C^n_{NOT} involves $(n-1)$ qubits as control and the nth qubit as target. The $f - C_{NOT}$ (called function-CNOT) gate inverts the target state on the condition that the control states satisfy a specific Boolean function. The $f - C_{NOT}$ could be used for *eigenvalue-kickback* effect to induce π phase shifts on some component states if the auxiliary state is initially prepared as $(|0\rangle - |1\rangle)/\sqrt{2}$ (see Appendix).

The logic operation AND is carried out by a single C^3_{NOT} [35], and implementation of OR consists of a single C^3_{NOT} and some NOT operations. To accomplish each of them, an auxiliary qubit, initially prepared in $|0\rangle$, is employed to store the output of each operation.

Using above-mentioned logical gates, we depict the quantum circuits in Fig. 2 for a complete Grover search to solve the 2-SAT problem in Eq. (2). We can find that, besides two qubits a and b encoding the two variables, respectively, four auxiliary qubits are required, where q_1 and q_2 are to store the results of $\bar{a} \vee \bar{b}$ and $a \vee b$, respectively. q_3 will record the result of F_2, and q_4 is the ancilla to do *eigenvalue-kickback*. The first part of the quantum circuit $\tilde{U_0}$ is for Oracle's job, including solution to F_2 and the SI operation $I_\tau = I - 2|\tau\rangle\langle\tau|$ using the effect of *eigenvalue-kickback*. It consists of following five steps: (i) gates 1 and 2 perform the first clause $(\bar{a} \vee \bar{b})$ in Eq.(2); (ii) gates 3, 4, and 5 implement the second clause $(a \vee b)$; (iii) the gate 6 is to restore the states of the qubits a and b; (iv) the gate 7 executes AND for three clauses to obtain the value of F_2, which achieves

$$(1/2\sqrt{2})(|00\rangle_{ab}|100\rangle_{q_1q_2q_3} + |01\rangle_{ab}|110\rangle_{q_1q_2q_3} - |10\rangle_{ab}|111\rangle_{q_1q_2q_3} + |11\rangle_{ab}|010\rangle_{q_1q_2q_3})(|0\rangle - |1\rangle)_{q_4}. \tag{3}$$

Therefore, the target state is $|10\rangle_{ab}|111\rangle_{q_1q_2q_3}(|0\rangle-|1\rangle)_{q_4}$, which implies the SAT solution to be $a=1$ and $b=0$. To efficiently single out the target state, we have to discard the auxiliary qubits q_{1-3}, but keep the phase flipping in the qubit subspace. To this end, we employ reverse operations in an additional circuit $\tilde{U}_{add}$ to do $\tilde{U}_0^{-1}$. After the gating $1^{'}$, we obtain $(1/2)(|00\rangle+|01\rangle-|10\rangle+|11\rangle)_{ab}|000\rangle_{q_1q_2q_3}(|0\rangle-|1\rangle)_{q_4}$. Consequently, all the auxiliary qubits are decoupled from the qubits. We may discard q_1, q_2 and q_3, but keep q_4 for later use.

The function of IAA performed in the subsequent dotted-box of the quantum circuit is to single out the target state. As mentioned previously [23, 33, 34], in the implementation of IAA, the diffusion transform $\hat{D}^{(n)}$ is always unchanged no matter which state is to be searched. As a result, we may design the IAA as a fixed module. Since the target state is labeled by SI operation, once IAA (i.e., gate 8) is carried out, we should be able to single out the target state.

Specifically, the operation in the step of IAA could be represented by

$$\hat{D}^{(2)}=-H^{\otimes 2}\sigma_{x,b}\sigma_{x,a}C_{PF}^{(2)}\sigma_{x,a}\sigma_{x,b}H^{\otimes 2}, \tag{4}$$

where $\hat{D}^{(2)}$ is the diffusion transform $\hat{D}^{(n)}$ in the case of two qubits, $C_{PF}^{(2)}$ =diag$\{1,1,1,-1\}$ in the subspace spanned by $\{|0\rangle_a,\, |1\rangle_a,\, |0\rangle_b,\, |1\rangle_b\}$ inverts the state $|11\rangle$ to be $-|11\rangle$,

$$H^{\otimes 2}=\prod\nolimits_{i=1}^{2}H_i=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \tag{5}$$

and $\sigma_{x,i}$ is the single-qubit NOT gate acting on the qubit i ($i=a$ or b).

With more variables, the accomplishment of our scheme for solution to K-SAT problem would be more complicated. For example, we have designed a quantum circuit in Fig. 3 for a 2-SAT formula with three clauses and three variables, i.e.,

$$F_2^{\prime}=(a\vee b)\wedge(\bar{a}\vee c)\wedge\bar{b}, \tag{6}$$

where a, b, and c denote different variables. Due to similarity to operations in Fig. 2, we will not discuss it in detail. In addition, we have also designed a quantum circuit in Fig. 4 for a 3-SAT problem with four clauses and three variables, which reads

$$F_3=(a\vee b\vee c)\wedge(a\vee\bar{b}\vee c)\wedge b\wedge\bar{c}. \tag{7}$$

It is evident that more operations are needed with more variables involved, and for more clauses, more auxiliary qubits are required. One thing we have to mention is that, Grover search for more than two qubits are carried out with probability. So to obtain the final result, we have to perform SI and IAA repeatedly. But for clarity, we have only plotted in Figs. 4 and 5 a single iteration.

IV. EXPERIMENTAL FEASIBILITY WITH TRAPPED IONS BY TWO METHODS

In the previous section, we have designed some quantum circuits for a complete Grover search to solve K-SAT problems, where the two-qubit C_{NOT}^2 gate and multiqubit C_{NOT}^n ($n\geq 3$) gate play crucial roles. To have efficient quantum operations, however, we expect less operations and shorter implementing time. So in contrast with conventional methods [36, 37] using a number of single-qubit and two-qubit gates to compose a C_{NOT}^n gate, a straightforward performance of C_{NOT}^n gate seems more attractive [31, 38]. In what follows, we will try to carry out those quantum circuits with ultracold trapped ions by two different methods for C_{NOT}^n gating: conventional way and straightforward way.

Let us first consider the conventional way by taking LS gates [30] as an example. In ion trap QC [24], the qubits are encoded in each ion's internal ground state $|\downarrow\rangle$ and excited state $|\uparrow\rangle$, and quantum gates are made via excitation of the common vibrational modes. We assume that the ions could be irradiated individually by lasers, and we have a string of n ions confined in a linear trap, with strong confinement along $\vec{x}$ and $\vec{y}$ directions, but less strong confinement along the $\vec{z}$ axis. According to [30], conditional on the resonance condition $\Omega_0=\omega_z/2$ with Ω_0 the Rabi frequency and ω_z the $\vec{z}$-axis vibrational frequency, the ac Stark shift induced by the laser in resonance with the carrier transition frequency allows the ionic internal state to couple ionic motional state in a way exactly analogous to the red detuning transition. As a result, the two-qubit C_{NOT}^2 gate could be achieved by sequences of laser pulses corresponding to following unitary operators on jth ion: $\hat{C}_j^{\{\vartheta,q\}}=\exp[-i\frac{\vartheta}{2}(|\uparrow\rangle_j\langle\downarrow|+|\downarrow\rangle_j\langle\uparrow|)]$, or $\hat{R}_j^{\{\vartheta^{'},q^{'}\}}=\exp[-i\frac{\vartheta^{'}}{2}(a\,|\uparrow\rangle_j\langle\downarrow|-a^{\dagger}\,|\downarrow\rangle_j\langle\uparrow|)]$, where $\vartheta=2\Omega_0 t$, $\vartheta^{'}=2\Omega_0\eta t$, a and $a^{\dagger}$ are the annihilation and creation operators of the quantized center-of-mass

(COM) mode, respectively, and $q(q^{'}) = I, II$ refers to laser polarization for achieving the transitions $|\downarrow\rangle \longrightarrow |\uparrow\rangle$ and $|\downarrow\rangle \longrightarrow |aux\rangle$ (See Fig. 5), respectively, with $|aux\rangle$ the auxiliary state defined in [24].

Concretely speaking, a two-qubit C_{NOT}^2 gate could be constructed by following laser sequence: $C_{NOT}^2 = \hat{R}_2^{\{\pi,I\}}\hat{C}_1^{\{\pi/2,II\}}\hat{R}_1^{\{2\pi,II\}}\hat{C}_1^{\{\pi/2,II\}}\hat{R}_2^{\{\pi,I\}}\hat{C}_1^{'}$ including four types of basic operators: (i) A gate: $\pi/2$ pulse for carrier transition, which implements single-qubit rotations; (ii) B gate: π pulse for red detuning transition, which changes the electronic state of the ion as well as the vibrational state of the ion string; (iii) B^* gate: 2π pulse for red detuning transition; (iv) A^* gate: the last local operation $\hat{C}_1^{'}$ in the laser sequence, used for removing the undesired phase acquired in previous steps. These four types of basic operators enable us to realize quantum gates between two arbitrary ions in an ion string mediated by the vibrational mode. Based on C_{NOT}^2, the multiqubit C_{NOT}^n gate can be constructed by the standard decomposition scheme [36], where the C_{NOT}^n gate (for $n = 3, 4, ..., 8$) consists of $(2^n - 2)$ C_{NOT}^2 gates and 2^n one-qubit gates. In other words, a C_{NOT}^n gate is constituted by $(2^{n+1} - 4)$ A gates, $(2^n - 2)$ A^* gates, $(2^{n+1} - 4)$ B gates, and $(2^n - 2)$ B^* gates, in addition to 2^n one-qubit gates. So realization of a C_{NOT}^n gate by the conventional way requires $(8 \times 2^n - 12)$ laser pulses.

Alternatively, for the straightforward way, we take our recent proposal [31] as an example, in which the qubit encoding in the last ion is different from that in other ions (See Fig. 5). By this way, we could have a one-step implementation of CPF gate $C_{PF}^{(n)}$ $(n \geq 2)$. According to the scheme in [31], both success rate and fidelity of the $C_{PF}^{(n)}$ gate depend on a parameter m which equals to $\Omega_{\max}^n/\Omega_{\max}^i$ with $\Omega_{\max}^k$ $(k = 1, 2, ..., n)$ the maximum Rabi frequency regarding kth ion by a laser. The nth ion is the last ion. It was shown in [31] that the smaller the value of m, the better the performance of $C_{PF}^{(n)}$, and the gate $C_{PF}^{(n)} = J_{11\cdots 1}^{(n)}$ =diag$\{1, \cdots, 1, -1\}$ is to make a phase flip on the nth ion, which yields the multiqubit C_{NOT}^n gating,

$$C_{NOT}^n = H_n C_{PF}^{(n)} H_n, \tag{8}$$

with H_n the single-qubit Hadamard gate acting on the nth ion. Note that the single-qubit operation takes negligible time in comparison with multiqubit $C_{PF}^{(n)}$ gate, so the C_{NOT}^n gating time is almost equal to the $C_{PF}^{(n)}$ gating time, and the realization of a C_{NOT}^n gate by a straightforward way requires only $(n + 2)$ individually addressing laser pulses.

In what follows, we compare the two above-mentioned methods working with trapped $^{40}Ca^+$ to carry out the quantum circuits in Figs. 2, 3, and 4. To have a good confinement of the trapped ions, we have fixed the Lamb-Dicke parameter to be $\eta = 0.02$ throughout our calculation, where $\eta = (\vec{k}\cos\theta)\sqrt{\hbar/2nM\omega_z}$ with $\vec{k}$ the wave vector of the laser, θ the angle of the laser radiation with respect to $\vec{z}$ axis, n the number of the ions, and M the mass of the ion. In our estimate, we have set the angle $\theta = 30°$. As a result, we have different axial trapping frequencies for different numbers of ions. For example, in quantum circuits I, II, and III, we have $\omega_z/2\pi = 2.92$ MHz, 2.50 MHz, and 1.94 MHz, respectively. Moreover, in both methods, we consider strong laser radiation, i.e., $\Omega_0 = \omega_z/2$ in the conventional way, and $\Omega_{\max}^i = \omega_z/2$ $(i \neq n)$ in the straightforward way. For the scheme with LB gates [30], the C_{NOT}^2 gating time is mainly determined by B and B^* gating time since other gates works much faster. So we may only consider the values of T_B and T_{B^*} to achieve B and B^* gates with $T_B = \pi/2\Omega_0\eta = 50\pi/\omega_z$, and $T_{B^*} = 2T_B$. In contrast, for our CPF proposal [31], in each circuit, once η is determined, the $C_{PF}^{(n)}$ gating time is irrelevant to the number of the qubits involved, but determined by the Rabi frequency regarding the last qubit, i.e., $\Omega_{\max}^n$. Like in [31], we set $\Omega_{\max}^n = \Omega_{\max}^i/10$, which yields the $C_{PF}^{(n)}$ gating time to be $T = \pi/\eta\Omega_{\max}^n = 1000\pi/\omega_z$. We have listed our results in Tab. I, where both methods take millisecond time-scale for an implementation of our scheme.

In addition, as $C_{PF}^{(n)}$ in the straightforward way has intrinsic success probability regarding $m = \Omega_{\max}^n/\Omega_{\max}^i$, we have also shown by numerical simulation the implementation of the Grover search for solution to K-SAT problem in Fig. 6, in which both the success probability of the $C_{PF}^{(n)}$ gate and the intrinsic probability of the Grover search itself are involved. The lower success rates for the three-qubit cases, i.e., the solid and dotted-dashed curves in Fig. 6, are due to the intrinsic probability of the Grover search. Actually, once we have finished the second iteration, the success rates would be much higher [34].

V. DISCUSSION

To discuss the experimental feasibility of our scheme, we consider ultracold trapped Calcium ions $^{40}Ca^+$ in a linear trap as an example. If we adopt the conventional way [30], the ground state $|\downarrow\rangle$ and excited state $|\uparrow\rangle$ could be encoded in $S_{1/2}$ $(m_j = -1/2)$ and $D_{5/2}$ $(m_j = -1/2)$, respectively, and $D_{5/2}$ $(m_j = -3/2)$ can be one of the candidates for the auxiliary state $|aux\rangle$. In contrast, once the straightforward way is utilized, as done in [31], for the first $(n - 1)$ ions, the qubits $|\downarrow\rangle$ and $|\uparrow\rangle$ are encoded in $S_{1/2}$ $(m_j = 1/2)$ and $S_{1/2}$ $(m_j = -1/2)$, namely, the Zeeman sublevels of the ground state $S_{1/2}$ [40], but for the last nth ion, the qubits $|\downarrow\rangle$ and $|\uparrow\rangle$ are encoded into $S_{1/2}$ $(m_j = 1/2)$ and $D_{5/2}$

($m_j = -1/2$), respectively. The prerequisite of the experiment includes the accurate tuning of the laser pulses to the desired frequencies and phases, and the initial preparation of the vibrational mode of the ions to the ground state.

Both the conventional way and the straightforward way include following common features: i) the COM motion is utilized as the data bus, namely, the quantum logic operations (except for the carrier transition) involving the degrees of freedom of the quantized motion. So the vibrational mode should be laser cooled to the ground state, and thereby heating becomes a dominant source of decoherence; ii) Individually addressing by lasers is required. As a result, the laser intensity fluctuation $\Delta\Omega_0$ and the phase fluctuation $\Delta\phi$ should be well controlled for achieving high fidelity of gating [22, 39].

As we know, heating time of the ground vibrational state of the ions in the linear trap is on timescale of millisec, e.g., longer than 4 millisec for multi-ion trapping in NIST experiments [41] and even as long as 190 millisec for a single trapped ion in Innsbruck experiments [42]. In contrast, the lifetime of the metastable level $D_{5/2}$ of $^{40}Ca^{+}$ is much long, i.e., for about 1.16 sec [43]. So our operational time should be restricted within tens of millisec. Fortunately, from our calculation, the required total time T regarding both methods are shorter than 10 millisec. Nevertheless, to have a completely heating-free operation, we may accelerate our manipulation by some other ways. One of the ways is to optimize the quantum circuits [44], i.e., minimizing the number of gates required by using geometric approach [45]. Alternatively, we may consider to improve the efficiency of quantum gates. For example, we may employ a tighter trap to enhance ω_z, or consider to exactly adjust the magic numbers regarding the Lamb-Dicke parameter [47].

On the other hand, we can also find advantages of the straightforward way over the conventional way. First, the less number of laser pulses makes operation easier in practice. In the conventional way [30], the number of laser pulses required to accomplish C^n_{NOT} gate is much larger than the counterpart required in the straightforward way. So overhead could be much reduced in the straightforward way to generate a multiqubit C^n_{NOT}. Secondly, the increase of the qubits in straightforward way [31] could improve the fidelity and the success probability of the $C^{(n)}_{PF}$ gate, which are favorable for a scalable Grover search. Thirdly, the reduction of the manipulation steps could diminish computational errors.

As shown in Tab. I, however, the straightforward way, although with much fewer operational steps, takes comparable time to the conventional way. The reason is that the $C^{(n)}_{PF}$ gating rate in straightforward way mainly depends on the value of the Rabi frequency $\Omega^n_{\max}$ regarding the nth ion by a laser, namely, to meet the condition $\mathrm{erf}[\eta\Omega^n_{\max}t_0/\sqrt{\pi}] \to 1$ [31]. On the other hand, to obtain a $C^{(n)}_{PF}$ gating with big enough fidelity and success probability, we require $m = \Omega^n_{\max}/\Omega^i_{\max} \ll 1$, which greatly restricts the value of $\Omega^n_{\max}$ and thereby limits the speed of $C^{(n)}_{PF}$ gate. As a result, in the few-qubit case, the straightforward way works only slightly faster than the conventional way, whereas it would be more and more efficient than the conventional way with more qubits involved. Furthermore, as operational overhead could be much reduced in the straightforward way, we prefer to use it for implementation of our scheme even in the few-qubit case.

Another point we should mention is that, the scheme in [31] is very sensitive to m. As shown in Fig. 6, with the value of m larger than 0.04, the laser intensity fluctuation and phase fluctuation would lead to considerable affect on implementation. To avoid this detrimental influence, we have to keep the laser in high stability, or we take the case with $m < 0.04$, which, however, would yield a longer gating time. So a trade-off would be taken in realistic implementation with the straightforward way.

For solving K-SAT problems with multiple solutions, we will need more gates to exclude the target states that have been found previously. As the design of the circuit for this job is straightforward and strongly relevant to the specific solution [12], we will not go further along this line in the present work. But more gates will yields longer time and higher requirement for implementation, which brings about more challenges.

The solution of K-SAT problems with more clauses needs more qubits. With more than two qubits involved, however, Grover search works only probabilistically. So several iterations are needed to accomplish a solution. On the other hand, with more ions confined, the vibrational mode spectrum becomes more and more complicated and the ions' spacing would be decreasing, which yields individually addressing of the ions to be more intricate. As a result, the extension of QC from a few qubits to a large number of qubits is quite technically challenging. Nevertheless, as eight ultracold trapped ions in entanglement have been experimentally available [27], our treatment in quantum circuits I, II, and III, which involves six, seven, and nine ions, respectively, looks very promising under currently achieved technology in trapped ion system.

VI. CONCLUSION AND ACKNOWLEDGMENT

It is still hard to find a practical quantum algorithm. That is why we have had few working quantum algorithms so far. As one of the most frequently mentioned quantum algorithms, however, Grover search had never been applied to any really practical problems, although it was mentioned to be useful for searching some personal information from a phone book. Most relevant proposals and experiments so far have neglected the Oracle's work. Actually when Oracle's job is involved in the implementation, much overhead would be needed, which makes the implementation of Grover search more complicated, but more realistic and practical.

The contribution of the present work lies in two aspects. First, simplified quantum circuits for solutions to K-SAT problems are designed. Compared to [12], we have less auxiliary qubits and less operations in our design to enable solutions to K-SAT problems by less than ten qubits, which is important in view of experimental realization. Secondly, we have explored the possibility to accomplish the operations required in the quantum circuits with trapped ions. Our discussion would be also helpful to apply our scheme to other QC candidate systems.

In summary, we have concentrated on the application of Grover search algorithm on some solvable K-SAT problems with few ultracold trapped ions. Specifically, we have designed quantum circuits to solve some 2-SAT and 3-SAT problems, and presented the feasibility, challenge, and possible efforts to accomplish the schemes with currently or near future available ion trap technology. We believe that our design could be further optimized and would be useful for exploring application of QC.

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VII. APPENDIX: EIGENVALUE KICKBACK EFFECT

Provided that we prepare the state of the target qubit q_c as $(|0\rangle - |1\rangle)/\sqrt{2}$ and the quantum state of the control qubits as $|\Psi\rangle_0 = \sum_{\rho\in[A']} c_\rho |\rho\rangle + \sum_{\upsilon\in[B']} c_\upsilon |\upsilon\rangle$, where $[A^{'}] = \{i \mid f(i) = 1,\ i \in S = (0, 1, \cdot\cdot\cdot, 2^{n-1})\}$ is the set of satisfied assignments and $[B^{'}] = S - [A^{'}]$ is the complementary set, the $f - C_{NOT}$ gate (Fig. 1(c)) will result in *eigenvalue-kickback,* which could effectively flip the phase of some components (i.e., $|\rho\rangle$) of the state $|\Psi\rangle_0$. This mechanism can be concretely explained as,

$$\begin{aligned}
&f - C_{NOT}\{(\sum\nolimits_{\rho\in[A']} c_\rho |\rho\rangle + \sum\nolimits_{\upsilon\in[B']} c_\upsilon |\upsilon\rangle) \otimes (|0\rangle - |1\rangle)/\sqrt{2}\} \\
&= \{\sum\nolimits_{\rho\in[A']} c_\rho |\rho\rangle \otimes NOT(|0\rangle - |1\rangle) + \sum\nolimits_{\upsilon\in[B']} c_\upsilon |\upsilon\rangle \otimes (|0\rangle - |1\rangle)\}/\sqrt{2} \\
&= \{\sum\nolimits_{\rho\in[A']} c_\rho |\rho\rangle \otimes (|1\rangle - |0\rangle) + \sum\nolimits_{\upsilon\in[B']} c_\upsilon |\upsilon\rangle \otimes (|0\rangle - |1\rangle)\}/\sqrt{2} \\
&= \{-\sum\nolimits_{\rho\in[A']} c_\rho |\rho\rangle + \sum\nolimits_{\upsilon\in[B']} c_\upsilon |\upsilon\rangle\} \otimes (|0\rangle - |1\rangle)/\sqrt{2}
\end{aligned}$$

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Captions of Figures

FIG. 1. The required basic logic gates, where • denotes the control qubit and $\oplus$ the target qubit. (a) Two-qubit controlled-NOT gate C^2_{NOT}, where qubit 1 and 2 denote control and target qubits, respectively; (b) multiqubit controlled-NOT gate C^n_{NOT} with $n-1$ control qubits $(q_1, \cdots, q_{n-1})$; (c) the $f-C_{NOT}$ (function-Controlled-NOT) gate, where the Boolean expression $f(q_1; \cdots; q_{n-1})$ can be satisfied by one or more than one Boolean variable assignment; (d) AND gate; (e) OR gate.

FIG. 2. Quantum circuit for Grover search to solve a 2-SAT problem in Eq. (2), where H denotes Hadamard gate, and the three auxiliary qubits q_{1-3} return to their initial states $|0\rangle$ after the operations in blocks $\tilde{U}_0$ and $\tilde{U}_{add}$.

FIG. 3. Quantum circuit for one iteration in Grover search to solve the 2-SAT problem in Eq. (6), where the dots mean omitted operations SI and IAA.

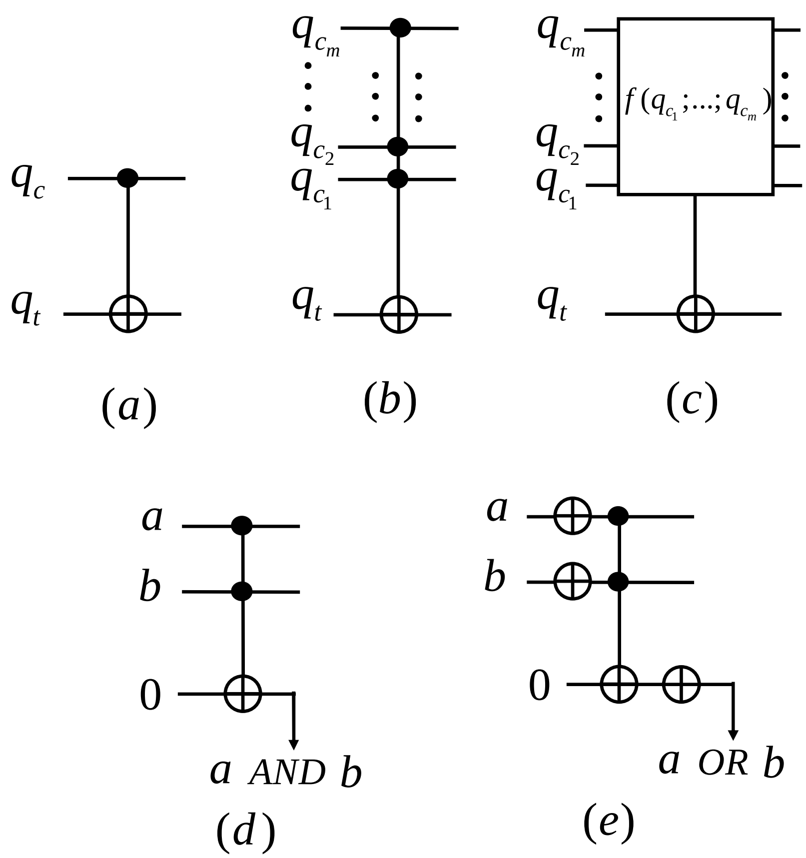
FIG. 4. Quantum circuit for one iteration in Grover search to solve the 3-SAT problem in Eq. (7), where the dots mean omitted operations SI and IAA.

FIG. 5. Schematic setup for implementing Grover search in a linear trap, where n ions are individually addressed by n lasers. The left inset shows the ionic-level configuration in LB gate [30], where the bold lines mean the levels encoding qubits, and the transition excited by the laser pulse depends on the laser polarization. The right inset shows the ionic level configuration in [31], where the qubit encoding in nth ion is different from in other ions, as labeled by the bold lines.

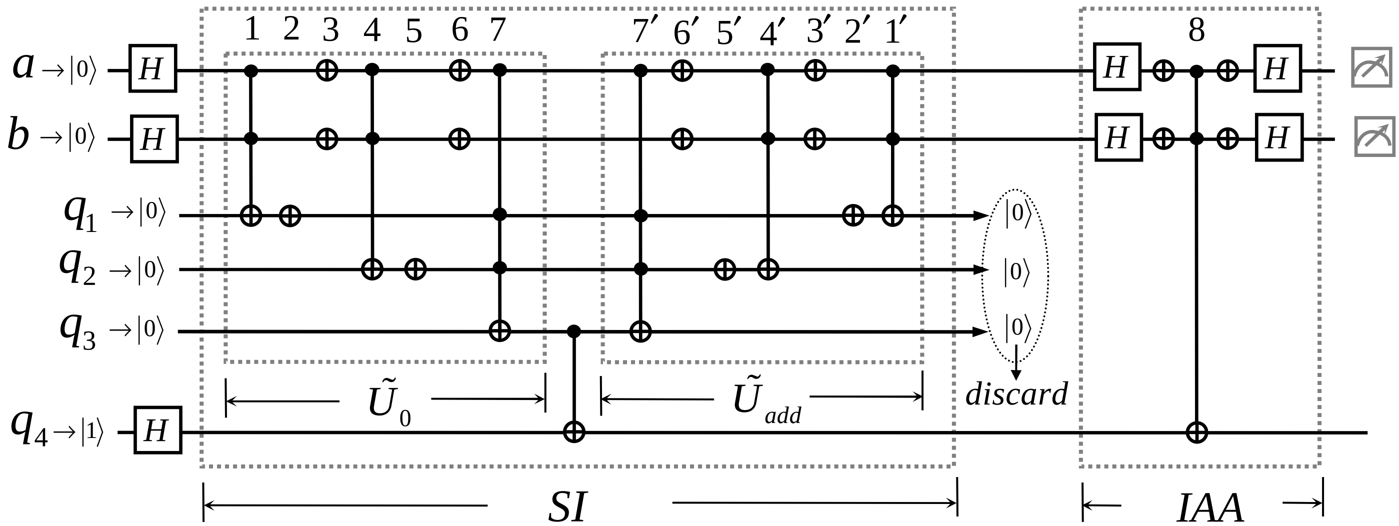
FIG. 6. The search probability for finding the solution to a K-SAT problem in one iteration in Grover search versus the parameter m, where the dotted, solid, dashed-dotted curves represent the cases of the quantum circuits I, II, and III, respectively.

TABLE I. Required time for implementation of one iteration in Grover search on $^{40}Ca^{+}$ by the quantum circuits I (Fig. 2), II (Fig. 3), and III (Fig. 4) using two different methods based on the LS gates [30] (called conventional way) and multiqubit CPF gate [31] (called straightforward way). In the conventional way, the total time T is the summation of the operational time for B, and B^* gate, specifically, $T = \{N[B] + 2N[B^*]\}T_B$, with $N[B]$, and $N[B^*]$ the number of the gates B, and B^* in the circuits, respectively, and T_B the time for B gating. In the straightforward way, $C_{PF}^{(i)}$ gating time is irrelevant to the number of the qubits involved, and $N[C_{PF}^{(i)}]$ denotes the total number of $C_{PF}^{(i)}$ required in the circuits with $i = 2, 3, 4, 5$ being the number of the qubits, and we have set $m = 0.1$.

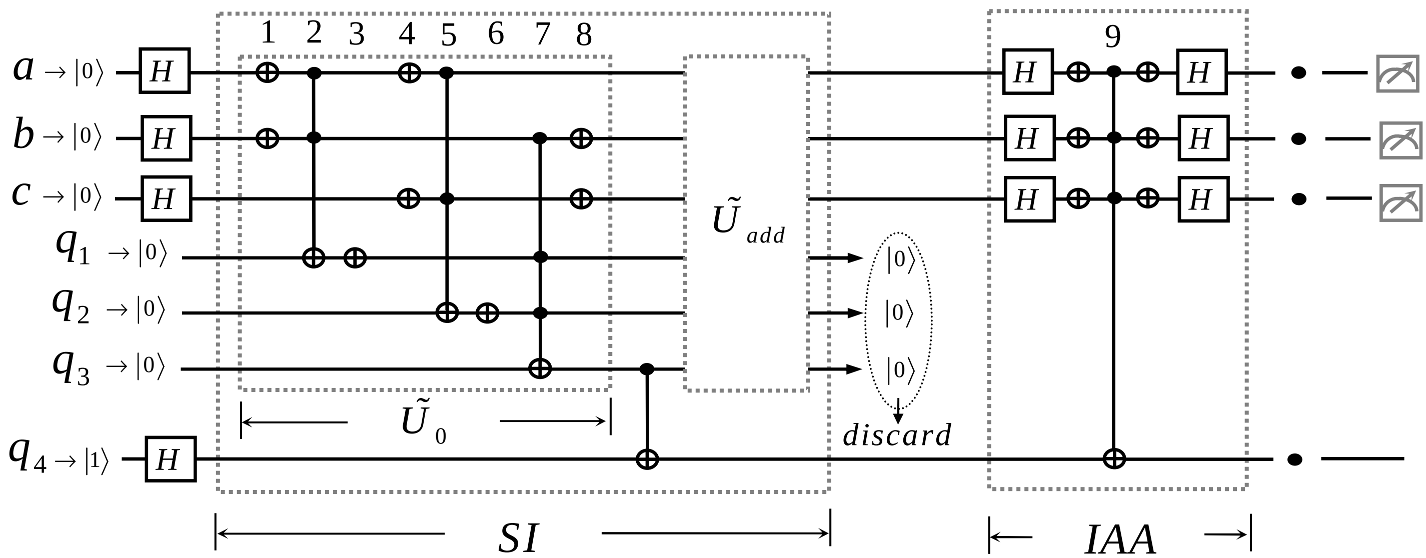
	$\vec{z}$-axis trap frequency	Conventional Way				Straightforward Way				
	$\omega_z/2\pi$ (MHz)	$N[B]$	$N[B^*]$	T_B (μs)	T (ms)	$N[C_{PF}^{(2)}]$	$N[C_{PF}^{(3)}]$	$N[C_{PF}^{(4)}]$	$N[C_{PF}^{(5)}]$	T (ms)
Circuit I	2.92	118	59	8.562	2.021	1	5	2	0	1.370
Circuit II	2.50	134	67	10.0	2.680	1	4	3	0	1.600
Circuit III	1.94	246	123	12.887	6.340	1	8	1	2	3.093



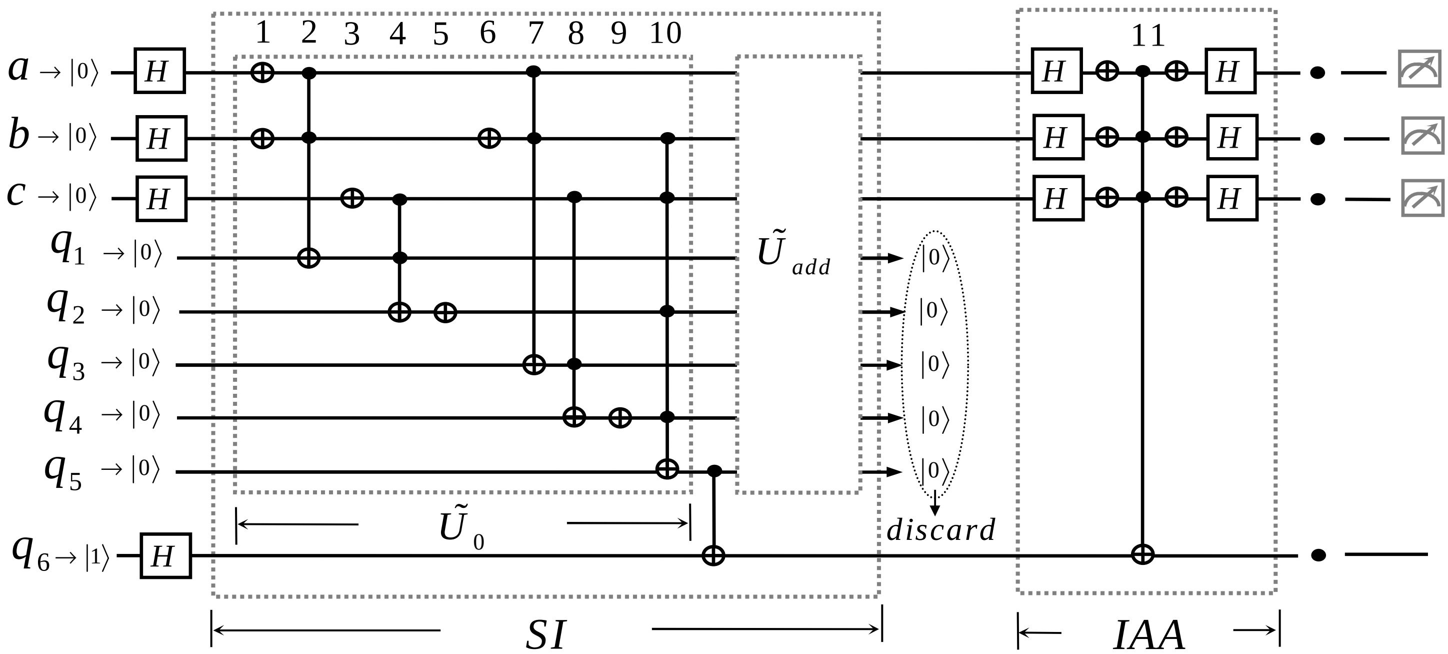
q_c
q_t
(a)
q_{c_m}
q_{c_2}
q_{c_1}
q_t
(b)
q_{c_m}
$f(q_{c_1};...;q_{c_m})$
q_{c_2}
q_{c_1}
q_t
(c)
a
b
0
a AND b
(d)
a
b
0
a OR b
(e)



1 2 3 4 5 6 7
7′ 6′ 5′ 4′ 3′ 2′ 1′
8
a → |0⟩
b → |0⟩
q1 → |0⟩
q2 → |0⟩
q3 → |0⟩
q4 → |1⟩
H
|0⟩
|0⟩
|0⟩
discard
Ũ0
Ũadd
SI
IAA



$a \rightarrow |0\rangle$
$b \rightarrow |0\rangle$
$c \rightarrow |0\rangle$
$q_1 \rightarrow |0\rangle$
$q_2 \rightarrow |0\rangle$
$q_3 \rightarrow |0\rangle$
$q_4 \rightarrow |1\rangle$
H
1 2 3 4 5 6 7 8
$\tilde{U}_0$
$\tilde{U}_{add}$
$|0\rangle$
discard
9
SI
IAA


1
2
3
4
5
6
7
8
9
10
11
a → |0⟩
b → |0⟩
c → |0⟩
q1 → |0⟩
q2 → |0⟩
q3 → |0⟩
q4 → |0⟩
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Ũ add
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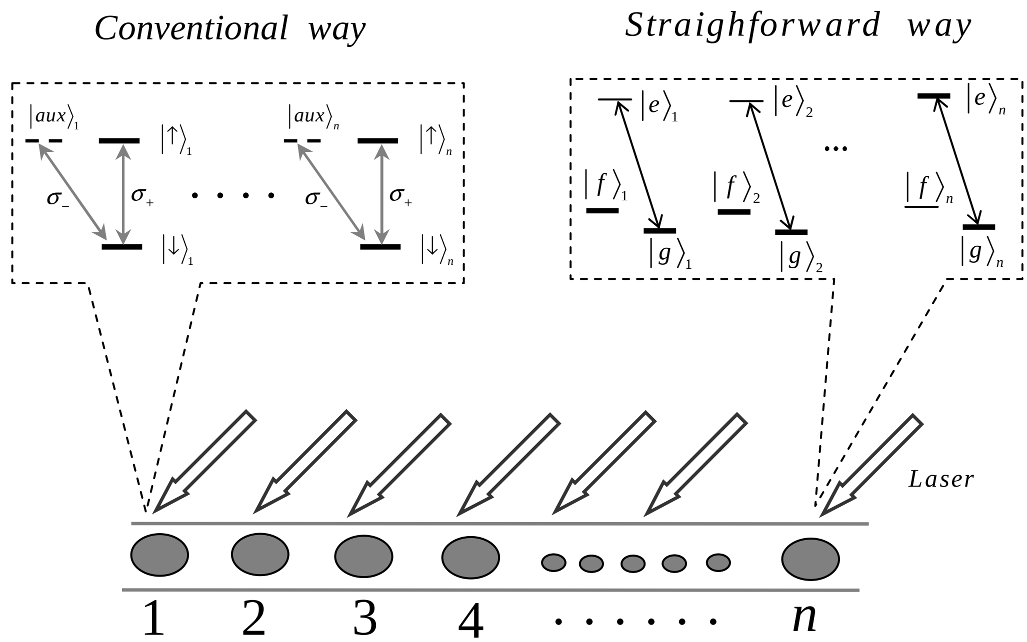
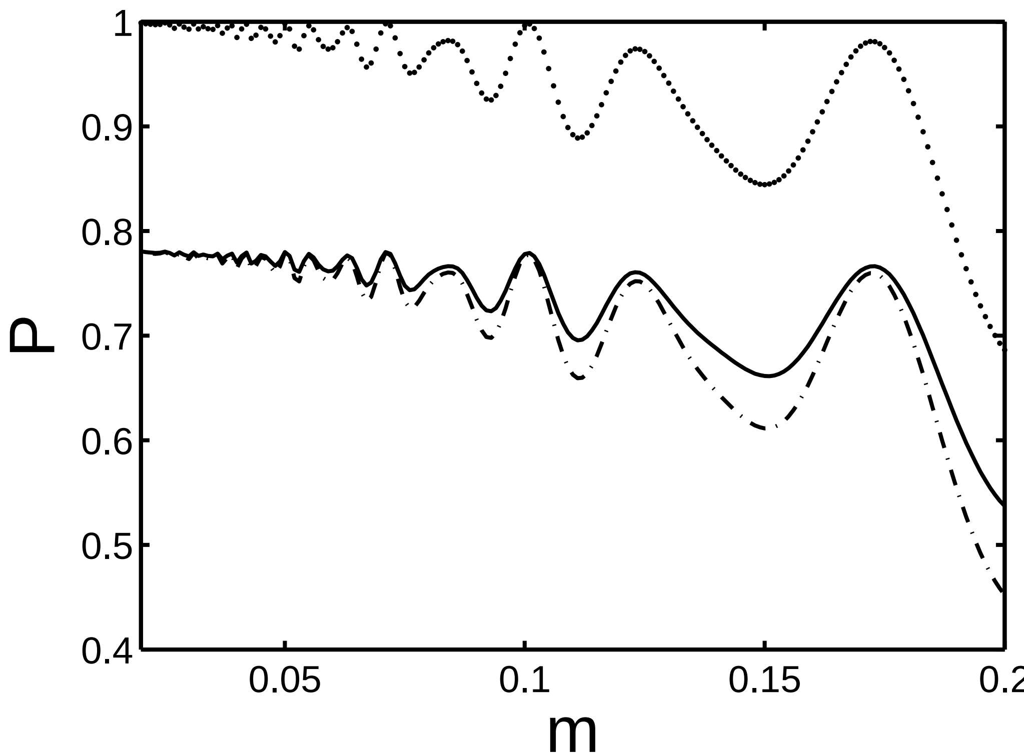
discard
Ũ 0
SI
IAA


Conventional way
Straighforward way
$|aux\rangle_1$
$|\uparrow\rangle_1$
σ_-
σ_+
$|\downarrow\rangle_1$
$|aux\rangle_n$
$|\uparrow\rangle_n$
$|\downarrow\rangle_n$
$|e\rangle_1$
$|f\rangle_1$
$|g\rangle_1$
$|e\rangle_2$
$|f\rangle_2$
$|g\rangle_2$
$|e\rangle_n$
$|f\rangle_n$
$|g\rangle_n$
Laser
1
2
3
4
n


1
0.9
0.8
0.7
0.6
0.5
0.4
P
0.05
0.1
0.15
0.2
m